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Reorientation of Three-Rotor Gyrostat under Uncontrolled External Disturbances

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Abstract

A nonlinear game problem of the three-axis reorientation of an asymmetric solid using three reaction wheels is considered for case of unconrolled external disturbances. We proposed an approach of reducing the above-mentioned nonlinear problem to a linear game theory problem.

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1 Introduction

The article studies the problem of three-axis reorientation of an asymmetric solid. Three reaction wheels are employed to produce necessary torque in the axes of the spacecraft. External uncontrolled disturbances, that have no statistical description, are taken into consideration in the process of reorientation.

The controlling moments, applied to flywheels, are offered to be generated by means of a feedback in form of nonlinear functions of phase variables of a considered conflict-controlled system of differential equations, which consists of dynamic Euler equations and kinematic equations based on the quaternion parametrization of attitude kinematics. It follows from [1,2] specific choice of the feedback structure for controlling moments is considered, which enables one to narrow down the nonlinear problem of reorientation to solving a linear game problems. The results found are the further development of the results [2]: 1) the algorithm for solving of the reorientation problem is extended to the case of an arbitrary initial and final angular positions of the solid; 2) the estimates of the accepted level of external disturbances are improved.

2 Statement of the problem of reorientation

Let us consider a mechanical system: an asymmetric solid and three homogeneous symmetric flywheels whose axes are aligned with the principal central axes of inertia of the solid. The angular motion of the system (gyrostat) under study with respect to the center of mass is described by the equations [1]

$$\begin{aligned}(A_1 - J_1)x_1' &= (A_2 - A_3)x_2x_3 + J_2x_3\varphi_2' - J_3x_2\varphi_3' - u_1 + v_1,\\ (A_2 - J_2)x_2' &= (A_3 - A_1)x_1x_3 + J_3x_1\varphi_3' - J_1x_3\varphi_1' - u_2 + v_2,\\ (A_3 - J_3)x_3' &= (A_1 - A_2)x_1x_2 + J_1x_2\varphi_1' - J_2x_1\varphi_2' - u_3 + v_3,\\ J_i(\varphi_i'' + x_i') &= u_i.\end{aligned} \tag{1}$$

Here x_i are the projections of the instantaneous angular velocity vector of the solid onto the principal central axes of inertia $\mathbf{k}_i$ of the gyrostat; A_i are the moments of inertia of the system with respect to the $\mathbf{k}_i$-axes; J_i, φ_i are the axial moments of inertia and the relative angular velocities of the flywheels. The controlling moments u_i applied to the flywheels. The moments v_i describe the moments of external forces and uncontrolled external disturbances.

Here and below i = 1, 2, 3; summation over the index i is from 1 to 3. Let us designate as $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\varphi}'$ the vectors consisting of x_i, u_i, v_i and φ_i', respectively.

Along with (1), let us consider the kinematic equations, which based on the quaternion parametrization of attitude kinematics

$$\begin{aligned}&2\eta_1' = \eta_4x_1 + \eta_2x_3 - \eta_3x_2, \quad 2\eta_2' = \eta_4x_2 + \eta_3x_1 - \eta_1x_3,\\ &2\eta_3' = \eta_4x_3 + \eta_1x_2 - \eta_2x_1, \quad \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1.\end{aligned} \tag{2}$$

We denote as $\boldsymbol{\eta}$ the vector, involving η_i and η_4, respectively.

Let us choose the feedback controlling moments) in the class functions $u_i = u_i(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\varphi}')$ with measurable realizations $u_i[t]$, which obey the constrains

$$|u_i| \le \alpha_i = \text{const} > 0. \tag{3}$$

Uncontrollable disturbances v_i can be realized in the form of any measurable functions $v_i = v_i[t]$ within the constraints

$$|v_i| \le \beta_i = \text{const} > 0. \tag{4}$$

Problem of reorientation. It is required to find for any admissible realizations of v_i the controlling moments u_i that transfer the solid from the initial state $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$ to the given state $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}_1$ during a finite time. Both that states are states of rest: $\mathbf{x}(t_0) = \mathbf{x}(t_1) = \mathbf{0}$. Also condition $\boldsymbol{\varphi}'(t_0) = \mathbf{0}$ takes place. The time instant $t_1 > t_0$ is not fixed. Without any loss of generality, we assume that $\boldsymbol{\eta}(t_1) = (0, 0, 0, 1)$.

3 An auxiliary linear conflict-control system

Let us consider the nonlinear controlling moments (only the expression for u_1 is presented, expressions for u_2 and u_3 are obtained from u_1 by the cyclic permutation if indices 1 → 2 → 3) [2]

$$\begin{aligned} u_1 = -\frac{2(A_1 - J_1)}{\eta_4}[u_1^*(\eta_1^2 + \eta_4^2) + u_2^*(\eta_1\eta_2 + \eta_3\eta_4) + u_3^*(\eta_1\eta_3 - \eta_2\eta_4) + \\ + 1/4\eta_1(x_1^2 + x_2^2 + x_3^2)] + (A_2x_2 + J_2\varphi_2')x_3 - (A_3x_3 + J_3\varphi_3')x_2 \end{aligned} \quad (5)$$

where u_i^* are some auxiliary controls.

An auxiliary linear conflict-controlled system can be isolated from closed nonlinear system (1), (2), (5):

$$\eta_i'' = u_i^* + v_i^*. \quad (6)$$

"Auxiliary disturbances" v_i^* have the form

$$v_1^* = 1/2\,[\eta_4 v_1/(A_1 - J_1) + \eta_2 v_3/(A_3 - J_3) - \eta_3 v_2/(A_2 - J_2)] \quad (1 \to 2 \to 3).$$

Using the Cauchy-Schwarz inequality and conditions (4), we deduce estimates

$$|v_i^*| \le \beta^* = 1/2[(\beta_1/(A_1 - J_1))^2 + (\beta_3/(A_3 - J_3))^2 + (\beta_2/(A_2 - J_2))^2]^{1/2}. \quad (7)$$

Let us now solve the problem of the fastest transfer of system (6) to the state

$$\eta_i = \eta_i' = 0 \quad (8)$$

for any admissible v_i^* which satisfy inequalities (7). We interpret this problem as a differential game in which one of the players has at its disposal the auxiliary control u_i^* and aims at decreasing time τ for taking the system (6) to the required position (8). The second player (opponent), whose objective is to increase τ (or to avoid attaining position (8) at all) has "auxiliary disturbances" v_i^* at his disposal.

For this problem to be solvable, the admissible levels u_i^* should be higher than the levels v_i^*. Let us represent corresponding constraints in the form

$$|u_i^*| \le \alpha_i^*, \; |v_i^*| \le \beta^* = \rho_i\alpha_i^*, \; 0 < \rho_i < 1. \quad (9)$$

The outlined game problem for system (6) is reduced [3] to the time-optimal problem for system

$$\eta_i'' = (1 - \rho_i)u_i^*. \quad (10)$$

Boundary conditions are the same as for system (6).

The solution of the time-optimal problem for system (10) has the form [4]

$$u_i^*(\eta_i, \eta_i') = \begin{cases} \alpha_i^* \operatorname{sgn} \psi_i^\rho(\eta_i, \eta_i'), & \psi_i^\rho \neq 0 \\ \alpha_i^* \operatorname{sgn} \eta_i = -\alpha_i^* \operatorname{sgn} \eta_i', & \psi_i^\rho = 0 \end{cases}. \qquad (11)$$

Here $\psi_i^\rho(\eta_i, \eta_i') = -\eta_i - [2(1-\rho_i)\alpha_i^*]^{-1}\eta_i'|\eta_i'|$ are the functions of switching.

Under $v_i^* \neq -\rho_i u_i^*$ the motion of system (6), (11) occurs initially (until the switching curves are reached) between the parabolic arc which are trajectories of systems $\eta_i'' = (1\pm\rho_i)u_i^*$, where u_i^* has the form (11). Next, once the switching curves $\psi_i^\rho(\eta_i, \eta_i') = 0$ are reached, system will move along these curves in the sliding mode until the desired state $\eta_i = \eta_i' = 0$ is attained.

The quantity

$$\tau = \max(\tau_i), \; \tau_i = 2\{|\eta_{i0}|(1-\rho_i)\alpha_i^*]^{-1}\}^{1/2} \qquad (12)$$

determines the minimum guaranteed time τ for attaining position $\eta_i = \eta_i' = 0$ in the auxiliary linear game-theoretic problem. Let us mention that those subsystems of the system being described that reach the necessary position earlier, than the last of them, will remain in this position. The relevant control u_i^* in those subsystems will be counteracted by “auxiliary disturbances” v_i^*.

4 The algorithm for solving the problem of reorientation

Solving the system related to η_i' in (2) as a set of algebraic equations for x_i, we arrive at the equalities

$$x_1 = \frac{2}{\eta_4}[\eta_1'(\eta_1^2 + \eta_4^2) + \eta_2'(\eta_1\eta_2 + \eta_3\eta_4) + \eta_3'(\eta_1\eta_3 - \eta_2\eta_4)] \quad (1 \to 2 \to 3). \qquad (13)$$

By the basis of (13) we conclude that transferring the system (6) to the position (8) in fact amounts to solving problem of reorientation during the finite time (12) by means of controlling moments (5).

The *iteration algorithm* of solving includes the following stages.

1 stage. Choosing construction (5) of controlling moments u_i in which u_i^* have the form (11). Construction (5) includes the multiplier $1/\eta_4$, which formally leads to the “singularity”. However, the analysis of phase trajectories of the system (6), (11) shows us that, in the case $\boldsymbol{\eta}(t_1) = (0, 0, 0, 1)$, the relation $|\eta_4| \in [|\eta_{40}|, 1]$ hold in the control process. Therefore, the mentioned “singularity” does not occur.

Let $\boldsymbol{\eta}(t_1) \neq (0, 0, 0, 1)$. In order to avoid a “singularity”, in this case it is sufficient to pass to the control moments obtained from (5) by permutation of indices (or to the combination of such control moments), i.e, along with (5) to apply the constructions of control moments of the form

$$u_i = \eta_s^{-1} f_i^{(s)}(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\varphi}', \mathbf{u}^*) \quad (s = \overline{1, 4}). \qquad (14)$$

The constructions (14) allow one at a certain choice of the functions $f_i^{(s)}$, to select linear auxiliary systems of type (6) from closed system of differential equations (1), (2), (14). In this case, the set of indices i in a system of type (6) will depend upon the index s of variable η in the denominator of expression (14). Thus, index $s = 4$ corresponds to $i = 1, 2, 3$, index $s = 1$ corresponds to $i = 2, 3, 4$, etc.

2 stage. Estimating levels β^* of v_i^* by using formulas (7).

3 stage. “Assigning” levels α_i^* of the auxiliary controls u_i^*. Numbers α_i^*, β^* predetermine the value of the guaranteed reorientation time $\tau = t_1 - t_0$.

4 stage. Verifying the constraints (3) imposed on controlling moments u_i. To this end we employ equalities (13). As result, this verification is possible along trajectories of linear system (6), (11) and also linear non-homogeneous system of differential equations for φ_i'. If estimates (3) fail to hold or, vice versa, they are satisfied with a kind of “reserve”, the search for the appropriate numbers α_i^* is to be continued. Otherwise, the reorientation time equals τ.

Taking into consideration the structure of controlling moments (5), we come to a conclusion, that for their estimates it is enough to have the estimates of $A_i x_i + J_i \varphi_i'$. This estimates have the form [2]

$$\left| A_i x_i(t) + J_i \varphi_i'(t) \right| \le t\left(\beta_1^2 + \beta_2^2 + \beta_3^2\right)^{1/2}. \tag{15}$$

5 Estimation of the admissible domain of disturbances

Let us find sufficient conditions for α_i, β_i, which determine the possibility of solving the problem of reorientation on the basis of the controlling moments (5).

T h e o r e m . *Let the admissible domain of the disturbances v_i be estimated by the inequalities*

$$\sqrt{3}\left(A_1 - J_1\right)\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2}\left[\sum\left(\beta_i^2/\left(A_i - J_i\right)^2\right)\right]^{1/2}$$
$$+ 4\sqrt{2}[2 + (\eta_{20}^2 + \eta_{30}^2)/\eta_{40}^2]^{1/2}\left(1 - \eta_{40}^2\right)^{1/2}\left[\sum \beta_i^2\right]^{1/2} < \alpha_1 \tag{16}$$
$$(1 \to 2 \to 3).$$

Then the problem of reorientation is solved by controlling moments (5), (11) which comply with given constraints (3).

Presented estimates has a feature of sufficiency, because employed inequalities were overstated. This estimates guarantees the solution of the problem of reorientation, when the value of τ is sufficiently large (although finite). If conditions (16) hold with a “reserve”, then the presented iteration algorithm can be used in order to determine the guaranteed time τ.

6 Proof of the theorem

For the sake of convenience let us represent the expressions (5) for moments u_i as a sum of two items : $u_i = u_i^{(1)} + u_i^{(2)}$, where

$$u_i^{(2)} = -\frac{2(A_i - J_i)}{\eta_4}[1/4\eta_i(x_1^2 + x_2^2 + x_3^2)].$$

Based upon the analysis of the phase portrait, we come to a conclusion, that for the possible states of the system (6), (11) the following inequalities take place

$$\eta_i^2 \le \eta_{i0}^2,\ \eta_{40}^2 \le \eta_4^2, \quad \max(\eta_i')^2 = |\eta_{i0}|[(\alpha_i^*)^2 - (\beta_i^*)^2]/\alpha_i^*.$$

Taking into account the equalities (13), the estimate (15) as well as the values (12) for τ_i, taking into account the phase portrait of the system (6), (11), using Cauchy-Schwarz inequality we get the following inequalities

$$\left|(A_2x_2 + J_2\varphi_2')x_3\right| \le \frac{2}{|\eta_4|}\left[\sum\beta_i^2\right]^{1/2}\times$$

$$\times\left|\tau_3\eta_3'(\eta_3^2+\eta_4^2) + \tau_1\eta_1'(\eta_3\eta_1 + \eta_2\eta_4) + \tau_2\eta_2'(\eta_3\eta_2 - \eta_1\eta_4)\right| \le$$

$$\le \frac{2}{|\eta_4|}\left(\eta_3^2+\eta_4^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2}\left[\sum\left(\tau_i\eta_i'\right)^2\right]^{1/2} \le$$

$$\le 4\left(1+\eta_{30}^2/\eta_{40}^2\right)^{1/2}\left[\left(\alpha_3^*+\beta^*\right)\eta_{30}^2/\alpha_3^* + \left(\alpha_1^*+\beta^*\right)\eta_{10}^2/\alpha_1^* + \left(\alpha_2^*+\beta^*\right)\eta_{20}^2/\alpha_2^*\right]^{1/2}\times$$

$$\times\left[\sum\beta_i^2\right]^{1/2}\left[\sum\left(\eta_{i0}^2\right)\right]^{1/2} \le 4\sqrt{2}\left(1+\eta_{30}^2/\eta_{40}^2\right)^{1/2}\left(1-\eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2}$$

$$(1 \to 2 \to 3).$$

Then, using the inequalities $\beta_i^* < \alpha_i^*$ and Cauchy-Schwarz inequality we get the following estimates

$$\left|u_1^{(1)}\right| \le 2\left(A_1 - J_1\right)\left(1+\eta_{10}^2/\eta_{40}^2\right)^{1/2}\left[\sum\alpha_i^{*2}\right]^{1/2}$$

$$+\ 4\sqrt{2}[2+\left(\eta_{20}^2 + \eta_{30}^2/\eta_{40}^2\right)]^{1/2}\left(1-\eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2} \quad (1 \to 2 \to 3).$$

In case $\alpha_i^* - \beta_i^* = \varepsilon > 0$ (where ε is a sufficiently small number), values of $|u_i^{(2)}|$ will be arbitrarily small. As a result, based upon conditions (3) we have

$$\sqrt{3}\left(A_1 - J_1\right)\left(1+\eta_{10}^2/\eta_{40}^2\right)^{1/2}\beta^*$$

$$+\ 4\sqrt{2}\left[2+(\eta_{20}^2+\eta_{30}^2)/\eta_{40}^2\right]^{1/2}\left(1-\eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2} \le \alpha_1 \quad (1 \to 2 \to 3).$$

Using them, taking the inequalities (7) into account, we have the estimates (16).

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